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does not involve any invariant subgroup besides the identity had been used by C. Jordan about twenty years earlier.

No one else seems to have adopted this later definition of simple group. Its implicit use in L. E. Dickson's *History of the Theory of Numbers,* volume 1, page 131, is doubtless due to an oversight. Curiously H. W. L. Tanner continued the use of his unfortunate definition in an article published about seven years later in volume 27 of the same journal, page 331.

The mathematical literature contains numerous definitions due to a lack of knowledge on the part of their authors. The different fates of the two cited above may serve to illustrate how difficult it is to deal with some of these definitions. The fact that they are potential sources of error, especially when they appear in periodicals which are frequently used, makes it undesirable to ignore even those which are not adopted by others.

QUESTIONS AND DISCUSSIONS.

EDITED BY W. A. HURWITZ, Cornell University, Ithaca, N. Y.

REPLIES.

36. A number of Discussions have been published in this department relating to cubic and biquadratic equations (cf. Vol. XXXV, p. 29, pp. 268–269 and 343–347; Vol. XXIV, pp. 136–137 and 436–439; Vol. XXIII, pp. 314–315). Below are given a number of questions sent in by Professor Harris Hancock of the University of Cincinnati which relate to the cubic and biquadratic and might, perhaps, more properly be proposed as problems were it not for the advantage to be gained, if possible, by treating them all, or at least several of them, in one discussion.

1. For what values of n can $\cos 2\pi/n$ be expressed in the form $(a + \sqrt{b})/c$ where a, b and c are integers?

2. Write the biquadratic in the form $ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0$. Show that its reducing cubic may be expressed by means of a determinant of the third order which when expanded is

$$4y^3 - g_2y - g_3 = 0,$$

where $g_2 = ae - 4bd + 3c^2$, and

$$g_3 = \begin{vmatrix} a, b, c \\ b, c, d \\ c, d, e \end{vmatrix} = ace - ad^2 - eb^2 - c^3 + 2bcd.$$

3. For the same biquadratic show that

$$a^3(x_0 + x_1 - x_2 - x_3)(x_0 + x_2 - x_1 - x_3)(x_0 + x_3 - x_1 - x_2) = 32(3abc - a^2d - 2b^3)$$

without making any use of symmetric functions, where $x_0, \cdots, x_3$ are the roots of the biquadratic equation.

4. If x_0, x_1, x_2 are the roots of a cubic and D its discriminant, show that x_1 is a rational function of x_0 and $\sqrt{D}$. Derive a much simpler relation than that given in Serret, *Cours d'Algèbre Supérieure,* 5th ed., No. 511.

5. If x_0, x_1, x_2, x_3 are the roots of a biquadratic, D its discriminant, e_1, e_2, e_3 the roots of the reducing cubic, show that x_1 is a rational function of x_0, e_1, e_2, e_3 and consequently also of x_0, e_1, and $\sqrt{D}$.

6. If the biquadratic

$$ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0 \qquad (1)$$

has a double root, show that the reducing cubic (cf., for example, Burnside and Panton's *Theory of Equations*)

$$t^3 + 3Ht^2 + \left(3H^2 - \frac{a^2I}{4}\right)t - \frac{G^2}{4} = 0 \tag{2}$$

has a root in common with the cubic

$$8t^3 + 12Ht^2 + G^2 = 0, \qquad (G \neq 0); \tag{3}$$

and conversely, if (2) and (3) have a common root, then (1) has a double root.

REPLY BY HARRIS HANCOCK, University of Cincinnati.

The following reply deals with parts 2–6 of Question 36.

2. With Descartes write the biquadratic $f(x) = ax^4 + 4bx^3 + 6cx^2 + 4dx + e$ in the form $(px^2 + 2qx + r)(p'x^2 + 2q'x + r')$, where

$$pp' = a, \qquad pq' + qp' = 2b, \qquad pr' + rp' + 4qq' = 6c, \qquad qr' + rq' = 2d, \qquad rr' = e.$$

The third of these expressions becomes $pr' + rp' = 2c - 4s$, if we put $qq' = c + s$. Note that the determinant

$$\begin{vmatrix} 2pp', & pq' + qp', & pr' + rp' \\ qp' + q'p, & 2qq', & qr' + rq' \\ rp' + r'p, & rq' + r'q, & 2rr' \end{vmatrix}$$

is the product

$$\begin{vmatrix} p, & p', & 0 \\ q, & q', & 0 \\ r, & r', & 0 \end{vmatrix} \begin{vmatrix} p', & p, & 0 \\ q', & q, & 0 \\ r', & r, & 0 \end{vmatrix} = 0.$$

Substituting the above values in the first determinant, we have

$$\begin{vmatrix} a, & b, & c - 2s \\ b, & c + s, & d \\ c - 2s, & d, & e \end{vmatrix} = 0;$$

or,

$$4s^3 - g_2 s - g_3 = 0,$$

where

$$g_2 = ae - 4bd + 3c^2,$$

and

$$g_3 = \begin{vmatrix} a, & b, & c \\ b, & c, & d \\ c, & d, & e \end{vmatrix}.$$

This form of the reducing cubic is well known. It is very important in the Theory of Elliptic Integrals and should be given in every text-book on the subject.

3. Let x_0, x_1, x_2, x_3 be the roots of the above quartic. It follows that

$$\frac{f(x) - f(x_0)}{x - x_0} = a(x - x_1)(x - x_2)(x - x_3).$$

Writing

$$x = -x_0 - \frac{2b}{a} = \tfrac{1}{2}(x_1 + x_2 + x_3 - x_0),$$

this expression becomes

$$\frac{f\left(-x_0 - \dfrac{2b}{a}\right) - f(x_0)}{2x_0 + \dfrac{2b}{a}} = \frac{a}{8}P,$$

where

$$P = (x_0 + x_1 - x_2 - x_3)(x_0 + x_2 - x_1 - x_3)(x_0 + x_3 - x_1 - x_2).$$

Note that P remains unchanged when we interchange x_0 with x_1, or with x_2, or with x_3. It follows that the equation of the third degree

$$\frac{f\left(-x-\dfrac{2b}{a}\right)-f(x)}{ax+b}=\tfrac{1}{4}P,$$

holds for four values of x, and consequently for every value of x. Hence, writing $x = 0$, it is seen that

$$\frac{8}{a^3}(3abc-a^2d-2b^3)=\tfrac{1}{4}P.$$

In a similar manner it may be proved that

$$f'(x_0)+f'(x_1)+f'(x_2)+f'(x_3)=P.$$

4. Write the cubic in the form

$$K(x) = x^3-p_1x^2+p_2x+p_3 \equiv (x-x_0)(x-x_1)(x-x_2)=0,$$
$$K'(x_0)=(x_0-x_1)(x_0-x_2),$$
$$\sqrt{D}=(x_1-x_0)(x_2-x_0)(x_2-x_1).$$

It follows that

$$x_2-x_1=\frac{\sqrt{D}}{K'(x_0)};$$

and since $x_2+x_1=p_1-x_0$, we have

$$2x_2=p_1-x_0+\frac{\sqrt{D}}{K'(x_0)},\qquad 2x_1=p_1-x_0-\frac{\sqrt{D}}{K'(x_0)}.$$

Similarly[1] it is seen that

$$2x_2=p_1-x_1-\frac{\sqrt{D}}{K'(x_1)},\qquad 2x_0=p_1-x_1+\frac{\sqrt{D}}{K'(x_1)};$$

$$2x_0=p_1-x_2-\frac{\sqrt{D}}{K'(x_2)},\qquad 2x_1=p_1-x_2+\frac{\sqrt{D}}{K'(x_2)}.$$

5. In Burnside and Panton, *Theory of Equations*, fourth edition, p. 124, it is shown that the roots of the above biquadratic may be written in the form

$$ax_0+b=\sqrt{e_1}+\sqrt{e_2}+\sqrt{e_3}=z_0,\qquad ax_1+b=\sqrt{e_1}-\sqrt{e_2}-\sqrt{e_3}=z_1,$$
$$ax_2+b=\sqrt{e_2}-\sqrt{e_1}-\sqrt{e_3}=z_2,\qquad ax_3+b=\sqrt{e_3}-\sqrt{e_1}-\sqrt{e_2}=z_3,$$

where e_1, e_2, e_3 are the roots of Euler's equation

$$t^3+3Ht^2+\left(3H^2-\frac{a^2I}{4}\right)t-\frac{G^2}{4}=0.$$

Those that are acquainted with the theory of the Galois realms of rationality will observe that if the algebraic number x_0 is adjoined to the realm R of the coefficients, and if the norm of the resulting realm $R(x_0)$ is taken, a realm, say $G = R(x_0)\cdot R(x_1)\cdot R(x_2)\cdot R(x_3) = R(x_0, x_1, x_2, x_3) = R(x_0, x_1, x_2)$ is formed, which is of degree 4!. This realm contains all rational functions of $\sqrt{e_1}$, $\sqrt{e_2}$, $\sqrt{e_3}$ and consequently also all rational functions of e_1, e_2, e_3. Hence G contains as a divisor the realm A, where A is the norm of the conjugate realms $R(e_1)$, $R(e_2)$, $R(e_3)$, a realm which is of degree 3!. It follows that the realm G taken with respect to the realm A is of the fourth degree, and that x_0 is a primitive quantity in G/A. Proofs of these statements are given below without using the Galois theory.

The following results may be established at once: Write $\sigma = z_0 = \sqrt{e_1}+\sqrt{e_2}+\sqrt{e_3}$ and $\tau = z_1 = \sqrt{e_1}-\sqrt{e_2}-\sqrt{e_3}$. It is seen that $\sigma^2 = e_1+e_2+e_3+2(\sqrt{e_1}\sqrt{e_2}+\sqrt{e_1}\sqrt{e_3}+\sqrt{e_2}\sqrt{e_3})$ with corresponding values for σ^3, σ^4, $\sigma^3\tau$, etc. Noting that $e_1+e_2+e_3 = -3H$, and $\sqrt{e_1}\sqrt{e_2}\sqrt{e_3} = -G/2$, we have the *rational relations*

$$\tau^4+\tau^3\sigma+3H(\tau^2+\tau\sigma+6e_1)+G(3\sigma+5\tau)-2e_1(\tau^2+2\tau\sigma-6e_1)=0,$$
$$\sigma^4+\sigma^3\tau+3H(\sigma^2+\sigma\tau+6e_1)+G(3\tau+5\sigma)-2e_1(\sigma^2+2\sigma\tau-6e_1)=0.$$

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[1] These formulæ exhibit the rational expressions desired unless $D = 0$; if $D = 0$, they may be modified without difficulty.

If in this pair of relations we put $\tau = z_2$, then $\sigma = z_3$. Change e_1 to e_2, then $\tau = z_1$ when $\sigma = z_3$, and further $\tau = z_2$ when $\sigma = z_0$; change e_1 to e_3, then $\tau = z_1$ when $\sigma = z_2$ and $\tau = z_0$ when $\sigma = z_3$. The above relations may be written in the form

$$-\sigma = \frac{\tau^4 + \tau^2(3H - 2e_1) + 5G\tau + 18He_1 + 12e_1^2}{\tau^3 + \tau(3H - 4e_1) + 3G},$$

or

$$\frac{-\sigma - \tau}{2} = \frac{e_1\tau^2 + G\tau + 9He_1 + 6e_1^2}{\tau^3 + \tau(3H - 4e_1) + 3G} = A_0\tau^3 + A_1\tau^2 + A_2\tau + A_3.$$

To determine the coefficients A_0, A_1, A_2, A_3, interchange σ and τ, and we have

$$\frac{-\tau - \sigma}{2} = A_0\sigma^3 + A_1\sigma^2 + A_2\sigma + A_3.$$

Through subtraction, it follows that $0 = A_0(\sigma^3 - \tau^3) + A_1(\sigma^2 - \tau^2) + A_2(\sigma - \tau)$, or

$$0 = A_0(\sigma^2 + \sigma\tau + \tau^2) + A_1(\sigma + \tau) + A_2.$$

Writing for σ and τ their values it is seen that $0 = A_0(-3H + 2e_1 + 2\sqrt{e_2}\sqrt{e_3}) + A_1 2\sqrt{e_1} + A_2$. From this it follows that

$$A_1 e_1 = A_0\frac{G}{2} \qquad \text{and} \qquad A_2 = A_0(3H - 2e_1).$$

These values substituted in the expression above give

$$-\sqrt{e_1} = A_0\left(\sigma^3 + \frac{G}{2e_1}\sigma^2 + 3H\sigma - 2e_1\sigma\right) + A_3.$$

Writing for σ its value, it follows, if we put $D = 8e_1^3 + 12He_1^2 + G^2$, that

$$A_0 = \frac{2e_1^2}{D}, \qquad A_1 = \frac{Ge_1}{D}, \qquad A_2 = \frac{6He_1^2 - 4e_1^3}{D}, \qquad A_3 = \frac{3Ge_1(2e_1 + H)}{D}.$$

It is thus seen that

$$\frac{-\sigma - \tau}{2} = \frac{e_1\tau^2 + G\tau + 9He_1 + 6e_1^2}{\tau^3 + \tau(3H - 4e_1) + 3G}$$
$$= \frac{2e_1^2\tau^3 + Ge_1\tau^2 + (6He_1^2 - 4e_1^3)\tau + 3Ge_1(2e_1 + H)}{8e_1^3 + 12He_1^2 + G^2}.$$

We thus have the pair of *integral* relations

$$0 = \sigma(8e_1^3 + 12He_1^2 + G^2) + 4e_1^2\tau^3 + 2Ge_1\tau^2 + (24He_1^2 + G^2)\tau + 6Ge_1(2e_1 + H),$$
$$0 = \tau(8e_1^3 + 12He_1^2 + G^2) + 4e_1^2\sigma^3 + 2Ge_1\sigma^2 + (24He_1^2 + G^2)\sigma + 6Ge_1(2e_1 + H),$$

with five other pairs of relations which correspond to those which have already been indicated for the rational relations.

The completion of the reply to Part 5 will be postponed until after the consideration of Part 6.

6. Writing $x = 2t$, $\lambda = 12H^2 - a^2I$, the reducing cubic is $F(x) = x^3 + 6Hx^2 + \lambda x - 2G^2$. Further put $F_1(x) = x^3 + 3Hx^2 + G^2$. By the algorithm of the greatest common divisor, the remainder, say $F_2(x)$ which is had by dividing $F(x)$ by $F_1(x)$ is $F_2(x) = 3Hx^2 + \lambda x - 3G^2$. The remainder derived by dividing $F_1(x)$ by $F_2(x)$ is, say, $F_3(x) = Ax + B$, where

$$A = \frac{9HG^2 - 9\lambda H^2 + \lambda^2}{9H^2} = \frac{9HG^2 + \lambda(3H^2 - a^2I)}{9H^2},$$

and

$$B = \frac{12G^2H^2 - \lambda G^2}{3H^2} = \frac{G^2a^2I}{3H^2}.$$

The remainder obtained on dividing $F_2(x)$ by $F_3(x)$, expressed also in terms of $F(x)$ and $F_1(x)$ is

$$9H^2[3G^2A^2 + (\lambda A - 3BH)B] = F_1(x)[9H^2A^2 + (3HAx + \lambda A - 3BH)(3Hx + 18H^2 - \lambda)] + F(x)\Phi(x),$$

where as seen below it is unnecessary to write down the value of the polynomial $\Phi(x)$. The term on the left-hand side of this equation is $G^2a^6(27J^2 - I^3) = G^2a^6\Delta$, where Δ is the discriminant of the biquadratic under discussion. This proves the theorem proposed. It is of course understood that G^2 is *different* from zero.

To continue the discussion further under the preceding Question, write the right-hand side of the above expression in the form

$$F_1(x)[C_0x^2 + C_1x + C_2] + F(x)\Phi(x).$$

It may be shown that

$$\begin{aligned} C_0 &= 9HG^2 - 9\lambda H^2 + \lambda^2 = 9HG^2 + (3H^2 - a^2I)\lambda = (a^2I)^2 - 6H^2a^2I - 9Ha^3J, \\ C_1 &= 3[3a^3Ja^2I + 2G^2a^2I - H(a^2I)^2 - 18H^2a^3J] \\ &= 3[6H^2G^2 + \lambda(G^2 + 2H\lambda - 18H^3)] \\ &= 3[a^3Ja^2I + H(a^2I)^2 - 8H^3a^2I - 18H^2a^3J], \\ C_2 &= 9(G^4 - \lambda^2H^2) + \lambda^3 + 12G^2\lambda H - 18HG^2a^2I \\ &= a^6\Delta + 6G^2(3a^3J + Ha^2I). \end{aligned}$$

Writing $x = 2e_1$ in the expression above and noting that $F(2e_1) = 0$, it is seen that

$$\frac{1}{8e_1^3 + 12He_1^2 + G^2} = \frac{4C_0e_1^2 + 2C_1e_1 + C_2}{G^2a^6\Delta}.$$

Returning to the integral relations in the preceding Question it is seen that

$$\begin{aligned} G^2a^6\Delta\tau &= B_0\sigma^3 + B_1\sigma^2 + B_2\sigma + B_3, \\ G^2a^6\Delta\sigma &= B_0\tau^3 + B_1\tau^2 + B_2\tau + B_3, \end{aligned}$$

where the B's are integral functions of the second degree in e_1. The coefficients of these latter functions are integral in H, G^2 and I. Writing for σ, τ the values z_0, z_1, z_2, z_3 and interchanging e_1 with e_2 and e_3, we have here six pairs of integral relations corresponding to the rational relations indicated in 5.

The last part of 5 follows from the first part by use of 4.

NOTE. To Question 36, proposed by Professor Hancock, his own reply, printed above, is the only one as yet received. A reply to part 1 of the question will be welcome.—EDITOR.

DISCUSSIONS.

We present this month two discussions, both related more closely to secondary than to collegiate mathematics, but of considerable interest also to teachers of the latter. Professor Johnson, in advocating, with much reason, the early use of the complex number and its representation by a point of the plane, has made several comments on which there may well be differences of opinion. It is hoped that expressions on such questions will be forthcoming from readers.

Professor McClenon states without argument certain modes of approaching the teaching of logarithms. Many, perhaps most, teachers will be able to express a preference regarding these diverse methods, without hesitation.

I. The Complex Quantity in Elementary Algebra.

By W. Woolsey Johnson, U. S. Naval Academy.

It is the purpose of this note to advocate an earlier introduction, to the student of elementary algebra, of the geometrical interpretation of the complex quantity $a + bi$ (Argand's diagram) than seems usually to be made.